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# Existence of solutions for generalized vector quasi-equilibrium problems in abstract convex spaces with applications

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## Abstract

In this paper, several kinds of generalized vector quasi-equilibrium problems are introduced and studied in abstract convex spaces. Using the properties of  $\Gamma$ -convex and  $\mathcal{RC}$ -maps, some sufficient conditions are given to guarantee the existence of solutions in connection with these generalized vector quasi-equilibrium problems. As applications, some existence theorems of solutions for the generalized semi-infinite programs with vector quasi-equilibrium constraints are also given.

**Keywords:** abstract convex space; generalized vector quasi-equilibrium problem; generalized semi-infinite program; set-valued mapping; KKM mapping

## 1 Introduction

It is well known that the vector quasi-equilibrium problem is an important generalization of the vector equilibrium problem which provides a unified model for vector quasi-variational inequalities, vector quasi-complementarity problems, vector optimization problems and vector saddle point problems. In 2000, Fu [1] established the existence theorems for the generalized vector quasi-equilibrium problems and the set-valued vector equilibrium problems. In 2003, Ansari and Fabián [2] considered a generalized vector quasi-equilibrium problem with or without involving  $\Phi$ -condensing mappings and proved the existence of its solution in real topological vector spaces. In 2005, Li *et al.* [3] studied the existence of solutions for two classes of generalized vector quasi-equilibrium problems. Recently, Lin *et al.* [4] introduced and studied a class of generalized vector quasi-equilibrium problems involving pseudomonotonicity hemicontinuity mappings under different conditions in topological vector spaces. Lin *et al.* [5] proved the existence of equilibria for generalized abstract economy with a lower semicontinuous constraint correspondence and a fuzzy constraint correspondence defined on a noncompact/nonparacompact strategy set. They also considered a systems of generalized vector quasi-equilibrium problems in topological vector spaces. Very recently, Yang and Pu [6] studied the existence and essential components in connection with the set of solutions for the system of strong vector quasi-equilibrium problems. Fu and Wang [7] considered the generalized strong vector quasi-equilibrium problems with domination structure. On the other hand, Ding [8] studied the existence of solutions for generalized vector quasi-equilibrium problems in locally

$G$ -convex spaces. Balaj and Lin [9] investigated existence of solutions for the generalized equilibrium problems in  $G$ -convex spaces.

The abstract convex space, introduced by Park [10] in 2006, includes the convex subset of a topological vector space, the convex space, the  $H$ -space, and the  $G$ -convex space as special cases. Moreover, Park [11] investigated the property of the abstract convex spaces and showed some applications. Recently, several authors have focused on the studies concerned with the set-valued maps and optimization problems in abstract convex spaces with applications. For instance, Cho *et al.* [12] studied some coincidence theorems and minimax inequalities in abstract convex spaces. Yang *et al.* [13] proved some maximal element theorems for set-valued maps in abstract convex spaces with applications. Yang and Huang [14] gave some coincidence theorems for compact and noncompact  $\mathcal{RC}$ -maps in abstract convex spaces with applications. Lu and Hu [15] established a new collectively fixed point theorem in noncompact abstract convex spaces with applications to equilibria for generalized abstract economies. Park [16] gave some comments on fixed points, maximal elements, and equilibria of economies in abstract convex spaces. Yang and Huang [17] studied the existence of solutions for the generalized vector equilibrium problems in abstract convex spaces. At the end of the paper [17], Yang and Huang pointed out that it is an interesting and important work to study some types of generalized vector quasi-equilibrium problems with moving cones in topological spaces. To the best of our knowledge, it seems that there is no work concerned with the study of the generalized vector quasi-equilibrium problems in abstract convex spaces. Therefore, it is natural and interesting to study some generalized vector quasi-equilibrium problems in abstract convex spaces under some suitable conditions.

On the other hand, we know that semi-infinite programs are constrained optimization problems in which the number of decision variables is finite, but the number of constraints is infinite. Since John [18] initiated semi-infinite programming precisely to deduce important results about two such geometric problems: the problems of covering a compact body in finite dimensional spaces by the minimum-volume disk and the minimum-volume ellipsoid, many researchers have been investigated the theory, applications and methods for the semi-infinite programming (see, for example, [19–22]). As a generalization of semi-infinite programming, the generalized semi-infinite programming has been become a vivid field of active research in mathematical programming in recent years due to its important applications to numerous real-life problems such as Chebyshev approximation, design centering, robust optimization, optimal layout of an assembly line, time minimal control, and disjunctive optimization (see [23] and the references therein). Therefore, it is important and interesting to study the existence of solutions concerned with some generalized semi-infinite programs with vector quasi-equilibrium constraints in abstract convex spaces.

The main purpose of this paper is to study several classes of generalized vector quasi-equilibrium problems in abstract convex spaces with applications to generalized semi-infinite programs. We give some sufficient conditions to guarantee the existence of solutions for these generalized vector quasi-equilibrium problems in abstract convex spaces. As applications, we give some existence theorems of solutions for the generalized semi-infinite programs under suitable conditions.

## 2 Preliminaries

Let  $X, Y$  be two nonempty sets. A set-valued mapping  $T : X \rightrightarrows Y$  is a mapping from  $X$  into the power set  $2^Y$ . The inverse  $T^{-1}$  of  $T$  is the set-valued mapping from  $Y$  to  $X$  defined

by

$$T^{-1}(y) = \{x \in X : y \in T(x)\}.$$

An abstract convex space  $(X, D, \Gamma)$  consists of a nonempty set  $X$ , a nonempty set  $D$ , and a set-valued mapping  $\Gamma : \langle D \rangle \rightrightarrows X$  with nonempty values, where  $\langle D \rangle$  denotes the set of all nonempty finite subset of a set  $D$ . If for each  $A \in \langle D \rangle$  with the cardinality  $|A| = n + 1$ , there exists a continuous function  $\phi_A : \Delta_n \rightarrow \Gamma(A)$  such that  $J \in \langle A \rangle$  implies  $\phi_A(\Delta_J) \subseteq \Gamma(J)$ , where  $\Delta_n$  is the standard  $n$ -simplex and  $\Delta_J$  the face of  $\Delta_n$  corresponding to  $J \in \langle A \rangle$ , then the abstract convex space reduces to the  $G$ -convex space. Let  $\Gamma_A := \Gamma(A)$  for  $A \in \langle D \rangle$ . When  $D \subset X$ , the space is defined by  $(X \supseteq D, \Gamma)$ . In this case, a subset  $M$  of  $X$  is said to be  $\Gamma$ -convex if, for any  $A \in \langle M \cap D \rangle$ , we have  $\Gamma_A \subseteq M$ . In the case  $X = D$ , let  $(X, \Gamma) := (X, X, \Gamma)$ .

It is easy to see that any vector space  $Y$  is an abstract convex space with  $\Gamma := \text{co}$ , where  $\text{co}$  denotes the convex hull in the vector space  $Y$ . Next we give more examples as follows.

**Example 2.1** ([10]) Let  $E$  be a topological vector space with a neighborhood system  $\mathcal{V}$  of its origin. A subset  $X$  of  $E$  is said to be almost convex (see [24] for more details) if for any  $V \in \mathcal{V}$  and for any finite subset  $A = \{x_1, x_2, \dots, x_n\}$  of  $X$ , there exists a subset  $B = \{y_1, y_2, \dots, y_n\}$  of  $X$  such that  $y_i - x_i \in V$  for all  $i = 1, 2, \dots, n$  and  $\text{co} B \subset X$ . Let  $\Gamma_A = \text{co} B$  for any  $A \in \langle X \rangle$ . Then  $(X, \Gamma)$  is a  $G$ -convex space and hence an abstract convex space.

**Example 2.2** ([10]) Usually, a convex space  $(E, \mathcal{C})$  in the classical sense consists of a nonempty set  $E$  and a family  $\mathcal{C}$  of subsets of  $E$  such that  $E$  itself is an element of  $\mathcal{C}$  and  $\mathcal{C}$  is closed under arbitrary intersection. For any given subset  $X \subset E$ , the  $\mathcal{C}$ -convex hull of  $X$  is defined as by

$$\text{Co}_{\mathcal{C}} X = \bigcap \{Y \in \mathcal{C} : X \subset Y\}.$$

We say that  $X$  is  $\mathcal{C}$ -convex if  $X = \text{Co}_{\mathcal{C}} X$ . Consider the mapping  $\Gamma : \langle E \rangle \rightrightarrows E$  defined by  $\Gamma_A = \text{Co}_{\mathcal{C}} A$ . Then  $(E, \Gamma)$  is an abstract convex space.

**Example 2.3** Let  $(M, d)$  be a pseudo-metric space, that is,  $d : M \times M \rightarrow [0, +\infty)$  such that, for every  $x, y, z \in M$ ,

- (i)  $d(x, x) = 0$ ;
- (ii)  $d(x, y) = d(y, x)$ ;
- (iii)  $d(x, z) \leq d(x, y) + d(y, z)$ .

For any  $A \in \langle M \rangle$ , define a set-valued mapping  $\Gamma : \langle M \rangle \rightrightarrows M$  by

$$\Gamma_A = \Gamma(A) = \bigcap \{B : B \text{ is a closed ball containing } A\}.$$

Then it is easy to see that  $(M, \Gamma)$  is an abstract convex space.

As pointed out by Park [25], the abstract convex space includes many generalized convex spaces as special cases such as  $L$ -spaces, spaces having property (H), pseudo  $H$ -spaces,  $M$ -spaces,  $G$ - $H$ -spaces, another  $L$ -spaces,  $FC$ -spaces and others. Some more examples of the abstract convex space and comments on it can be found in the literature [10, 25, 26] and the references therein.

Let  $(X, \Gamma)$  be an abstract convex space and  $V$  be a real topological vector space. Let  $E$  be a nonempty subset of  $X$ . Assume that  $S : E \rightrightarrows E$  and  $B : E \rightrightarrows E$  are two set-valued mappings. Suppose that  $F : X \times X \times X \rightrightarrows V$  and  $C : X \rightrightarrows V$  are two set-valued mappings such that for each  $x \in X$ ,  $C(x)$  is a closed convex cone with  $\text{int } C(x) \neq \emptyset$ , here  $\text{int } C(x)$  denotes the interior of  $C(x)$ . In this paper, we will consider the following generalized vector quasi-equilibrium problems in abstract convex spaces.

- (GVQEP1) Find  $\tilde{x} \in E$  such that

$$\tilde{x} \in S(\tilde{x}) \quad \text{and} \quad F(\tilde{x}, y, z) \subseteq C(\tilde{x}), \quad \forall y \in S(\tilde{x}), \forall z \in B(\tilde{x}).$$

We would like to mention that (GVQEP1) was considered by Lin *et al.* [4] in topological vector spaces. When  $S(x) = B(x) = E$  for all  $x \in E$ , (GVQEP1) was considered by Yang and Huang [17] in abstract convex spaces and by Balaj and Lin [9] in  $G$ -convex spaces, respectively.

- (GVQEP2) Find  $\tilde{x} \in E$  and  $\tilde{z} \in B(\tilde{x})$  such that

$$\tilde{x} \in S(\tilde{x}) \quad \text{and} \quad F(\tilde{x}, y, \tilde{z}) \subseteq C(\tilde{x}), \quad \forall y \in S(\tilde{x}).$$

When  $C(x)$  was replaced by  $-C(x)$ , (GVQEP2) was considered by Li and Li [27] in topological vector spaces. If  $S(x) = E$  for all  $x \in E$ , then (GVQEP2) was investigated by Fu and Wang [7] in topological vector spaces.

- (GVQEP3) Find  $\tilde{x} \in E$  such that

$$\tilde{x} \in S(\tilde{x}) \quad \text{and} \quad F(\tilde{x}, y, z) \cap -\text{int } C(\tilde{x}) = \emptyset, \quad \forall y \in S(\tilde{x}), \forall z \in B(\tilde{x}).$$

We note that (GVQEP3) was considered by Lin *et al.* [4] in topological vector spaces. When  $S(x) = B(x) = E$  for all  $x \in E$ , (GVQEP3) was studied by Yang and Huang [17] in abstract convex spaces and by Balaj and Lin [9] in  $G$ -convex spaces, respectively.

- (GVQEP4) Find  $\tilde{x} \in E$  and  $\tilde{z} \in B(\tilde{x})$  such that

$$\tilde{x} \in S(\tilde{x}) \quad \text{and} \quad F(\tilde{x}, y, \tilde{z}) \cap -\text{int } C(\tilde{x}) = \emptyset, \quad \forall y \in S(\tilde{x}).$$

We note that (GVQEP4) was considered by Li and Li [27] in topological vector spaces.

- (GVQEP5) Find  $\tilde{x} \in E$  such that

$$\tilde{x} \in S(\tilde{x}) \quad \text{and} \quad F(\tilde{x}, y, z) \not\subseteq -\text{int } C(\tilde{x}), \quad \forall y \in S(\tilde{x}), \forall z \in B(\tilde{x}).$$

When  $S(x) = B(x) = E$  for all  $x \in E$ , (GVQEP5) was investigated by Yang and Huang [17] in abstract convex spaces and by Balaj and Lin [9] in  $G$ -convex spaces, respectively.

- (GVQEP6) Find  $\tilde{x} \in E$  and  $\tilde{z} \in B(\tilde{x})$  such that

$$\tilde{x} \in S(\tilde{x}) \quad \text{and} \quad F(\tilde{x}, y, \tilde{z}) \not\subseteq -\text{int } C(\tilde{x}), \quad \forall y \in S(\tilde{x}).$$

It is worth mentioning that (GVQEP6) was considered by Lin *et al.* [4], and Li and Li [27] in topological vector spaces, respectively. Moreover, some special cases of (GVQEP6) were considered by Ansari and Fabián [2] in topological vector spaces.

- (GVQEP7) Find  $\tilde{x} \in E$  such that

$$\tilde{x} \in S(\tilde{x}) \quad \text{and} \quad F(\tilde{x}, y, z) \cap C(\tilde{x}) \neq \emptyset, \quad \forall y \in S(\tilde{x}), \forall z \in B(\tilde{x}).$$

When  $S(x) = B(x) = E$  for all  $x \in E$ , (GVQEP7) was studied by Yang and Huang [17] in abstract convex spaces and by Balaj and Lin [9] in  $G$ -convex spaces, respectively.

- (GVQEP8) Find  $\tilde{x} \in E$  and  $\tilde{z} \in B(\tilde{x})$  such that

$$\tilde{x} \in S(\tilde{x}) \quad \text{and} \quad F(\tilde{x}, y, \tilde{z}) \cap C(\tilde{x}) \neq \emptyset, \quad \forall y \in S(\tilde{x}).$$

When  $S(x) = B(x) = E$  for all  $x \in E$ , (GVQEP8) was considered by Lin [28] in topological vector spaces.

We would like to point out that, for a suitable choice of the spaces  $E, X, V$  and the mappings  $S, B, F, C$ , one can obtain a number of well-known insights into the generalized vector quasi-equilibrium problem [2, 4, 5, 7, 8, 27], the generalized vector equilibrium problem [9, 17, 28], the vector equilibrium problem, and the vector variational inequality problem [29, 30] as special cases of the problems (GVQEP1)-(GVQEP8).

Furthermore, assume that  $h : X \rightrightarrows L$  is a set-valued mapping, where  $L$  is a real topological vector space ordered by a closed convex pointed cone  $H \subseteq L$  with  $\text{int} H \neq \emptyset$ . It is clear that the existence of solutions for problems (GVQEP1)-(GVQEP8) is closely analogous to the existence of solutions in connection with the following generalized semi-infinite programs with generalized vector quasi-equilibrium constraints:

- (GSIP1) Generalized semi-infinite program with constraint (GVQEP1):

$$\text{wMin}_H h(K),$$

where

$$K = \{x \in E : x \in S(x), F(x, y, z) \cap -\text{int} C(x) = \emptyset, \forall y \in S(x), \forall z \in B(x)\}.$$

When  $S(x) = B(x) = E$  for all  $x \in E$ , (GSIP1) was considered by Yang and Huang [17] in abstract convex spaces.

- (GSIP2) Generalized semi-infinite program with constraint (GVQEP2):

$$\text{wMin}_H h(K),$$

where

$$K = \{x \in E : x \in S(x), \exists z \in B(x), F(x, y, z) \subseteq C(x), \forall y \in S(x)\}.$$

Some special cases of (GSIP2) were considered by Lin [28] in topological vector spaces.

- (GSIP3) Generalized semi-infinite program with constraint (GVQEP3):

$$\text{wMin}_H h(K),$$

where

$$K = \{x \in E : x \in S(x), F(x, y, z) \cap -\text{int} C(x) = \emptyset, \forall y \in S(x), \forall z \in B(x)\}.$$

When  $S(x) = B(x) = E$  for all  $x \in E$ , (GSIP3) was studied by Yang and Huang [17] in abstract convex spaces.

- (GSIP4) Generalized semi-infinite program with constraint (GVQEP4):

$$\text{wMin}_H h(K),$$

where

$$K = \{x \in E : x \in S(x), \exists z \in B(x), F(x, y, z) \cap -\text{int } C(x) = \emptyset, \forall y \in S(x)\}.$$

We would like to mention that some special cases of (GSIP4) were studied by Lin [28] in topological vector spaces.

- (GSIP5) Generalized semi-infinite program with constraint (GVQEP5):

$$\text{wMin}_H h(K),$$

where

$$K = \{x \in E : x \in S(x), F(x, y, z) \not\subseteq -\text{int } C(x), \forall y \in S(x), \forall z \in B(x)\}.$$

When  $S(x) = B(x) = E$  for all  $x \in E$ , (GSIP5) was investigated by Yang and Huang [17] in abstract convex spaces.

- (GSIP6) Generalized semi-infinite program with constraint (GVQEP6):

$$\text{wMin}_H h(K),$$

where

$$K = \{x \in E : x \in S(x), \exists z \in B(x), F(x, y, z) \not\subseteq -\text{int } C(x), \forall y \in S(x)\}.$$

We note that some special cases of (GSIP6) were considered by Lin [28] in topological vector spaces.

- (GSIP7) Generalized semi-infinite program with constraint (GVQEP7):

$$\text{wMin}_H h(K),$$

where

$$K = \{x \in E : x \in S(x), F(x, y, z) \cap C(x) \neq \emptyset, \forall y \in S(x), \forall z \in B(x)\}.$$

When  $S(x) = B(x) = E$  for all  $x \in E$ , (GSIP7) was studied by Yang and Huang [17] in abstract convex spaces.

- (GSIP8) Generalized semi-infinite program with constraint (GVQEP8):

$$\text{wMin}_H h(K),$$

where

$$K = \{x \in E : x \in S(x), \exists z \in B(x), F(x, y, z) \cap C(x) \neq \emptyset, \forall y \in S(x)\}.$$

It is worth mentioning that (GSIP8) can be considered as a generalization of the generalized vector semi-infinite programming introduced and studied by Lin [28] in topological vector spaces.

In brief, for suitable choice of the spaces  $L, V, X, E$  and the mappings  $S, B, F, C, h$ , one can obtain a number of known the generalized semi-infinite program [17], the mathematical program with equilibrium constraint [19], the generalized semi-infinite program [23], the generalized vector semi-infinite programming [28], and the vector optimization problem [30–32] as special cases from the problems (GSIP1)-(GSIP8).

Now, we recall some useful definitions and lemmas as follows.

**Definition 2.1** Let  $K \subseteq V$  be a nonempty set and  $C \subseteq V$  be the closed convex pointed cone with  $\text{int } C \neq \emptyset$ . The set of all weak minimal points of  $K$  with respect to the ordering cone  $C$  is defined as

$$\text{wMin}_C(K) = \{x \in K : (x - K) \cap \text{int } C = \emptyset\}.$$

**Definition 2.2** Let  $(X, D, \Gamma)$  be an abstract convex space and  $Z$  be a set. For a set-valued mapping  $T : X \rightrightarrows Z$  with nonempty values, if a set-valued mapping  $G : D \rightrightarrows Z$  satisfies

$$T(\Gamma_N) \subseteq G(N) := \bigcup_{y \in N} G(y) \quad \text{for all } N \in \langle D \rangle,$$

then  $G$  is called a KKM mapping with respect to  $T$ . A KKM mapping  $G : D \rightrightarrows X$  is a KKM mapping with respect to the identity mapping  $I_X$ .

A set-valued mapping  $F : X \rightrightarrows Z$  is called to be a  $\mathfrak{RC}$ -map if, for any closed-valued KKM mapping  $G : D \rightrightarrows Z$  with respect to  $F$ , the family  $\{G(y)\}_{y \in D}$  has the finite intersection property. We denote

$$\mathfrak{RC}(X, Z) := \{F : F \text{ is } \mathfrak{RC}\text{-map}\}.$$

**Definition 2.3** ([33]) Let  $X$  and  $Y$  be two topological spaces. A set-valued mapping  $F : X \rightrightarrows Y$  is said to be

- (i) upper semicontinuous (u.s.c.) at  $x_0$  if for any open set  $V \supseteq F(x_0)$ , there is an open neighborhood  $O_{x_0}$  of  $x_0$  such that  $F(x') \subseteq V$  for each  $x' \in O_{x_0}$ ,
- (ii) lower semicontinuous (l.s.c.) at  $x_0$  if for any open set  $V \cap F(x_0) \neq \emptyset$ , there is an open neighborhood  $O_{x_0}$  of  $x_0$  such that  $F(x') \cap V \neq \emptyset$  for each  $x' \in O_{x_0}$ ,
- (iii) continuous at  $x_0$  if it is both upper and lower semicontinuous at  $x_0$ ,
- (iv) upper semicontinuous (lower semicontinuous or continuous) on  $X$  if it is upper semicontinuous (lower semicontinuous or continuous) at every  $x \in X$ ,
- (v) closed if and only if its graph  $\text{Graph}(F) := \{(x, y) \in X \times Y : y \in F(x)\}$  is closed.

**Lemma 2.1** ([34]) Let  $X$  and  $Y$  be two topological spaces and  $F : X \rightrightarrows Y$  a set-valued mapping.

- (i) If  $Y$  is compact, then  $F$  is closed if and only if it is upper semicontinuous,
- (ii) if  $X$  is a compact space and  $F$  is a u.s.c. mapping with compact values, then  $F(X)$  is a compact subset of  $Y$ .

**Lemma 2.2** ([35]) *Let  $X$  and  $Y$  be two topological spaces and  $F : X \rightrightarrows Y$  be upper semicontinuous and  $F(x)$  is compact. Then for any net  $\{x_\alpha\} \subset X$  with  $x_\alpha \rightarrow x$  and  $y_\alpha \in F(x_\alpha)$ , there exists a subnet  $\{y_\beta\} \subset y_\alpha$  such that  $y_\beta \rightarrow y \in F(x)$ .*

**Lemma 2.3** ([36]) *Let  $X$  and  $Y$  be two topological spaces and  $F : X \rightrightarrows Y$  be lower semicontinuous at  $x \in X$  if and only if for any  $y \in F(x)$  and any net  $\{x_\alpha\}$  with  $x_\alpha \rightarrow x$ , there is a net  $\{y_\alpha\}$  such that  $y_\alpha \in F(x_\alpha)$  and  $y_\alpha \rightarrow y$ .*

**Lemma 2.4** ([10]) *Let  $(X, D, \Gamma)$  be an abstract convex space,  $Z$  a set, and  $T : X \rightrightarrows Z$  a set-valued mapping. Then  $F \in \mathfrak{RC}(X, Z)$  if and only if any  $G : D \rightrightarrows Z$  satisfying*

- (i)  $G$  is closed-values;
- (ii)  $F(\Gamma_N) \subseteq G(N)$  for any  $N \in \langle D \rangle$ ,

*we have*

$$F(E) \bigcap \{G(y) : y \in N\} \neq \emptyset$$

*for each  $N \in \langle D \rangle$ .*

**Lemma 2.5** ([32]) *Assume that  $A$  is a nonempty compact subset of a real topological vector space  $V$  and  $D$  is a closed convex cone in  $V$  with  $D \neq V$ . Then one has  $\text{wMin}_D A \neq \emptyset$ .*

An abstract convex space with any topology is called an abstract convex topological space. In the rest of this paper, let  $(X, \Gamma)$  be an abstract convex Hausdorff topological space and  $E$  be a nonempty compact subset of  $X$ . Let  $V$  be a topological vector spaces. Assume that  $T : X \rightrightarrows X$ ,  $B : E \rightrightarrows E$ ,  $S : E \rightrightarrows E$ ,  $F : E \times E \times E \rightrightarrows V$  and  $Q : E \rightrightarrows V$  are five set-valued mappings. Let  $\rho$  be a binary relation on  $2^V$  and  $\rho^c$  be the complementary relation of  $\rho$ . Let  $\alpha$  be any of the quantifiers  $\forall, \exists$ , and  $\bar{\alpha}$  be the other of the quantifiers  $\forall, \exists$ .

### 3 Main results

In order to show the existence of solutions for the vector quasi-equilibrium problems (GVQEP1)-(GVQEP8), we first give the following general result.

**Theorem 3.1** *Suppose that the following conditions are satisfied:*

- (i)  $T \in \mathfrak{RC}(X, X)$ ;
- (ii) for each  $y \in E$ , the set  $\{x \in E : (\bar{\alpha})z \in B(x), \rho^c(F(x, y, z), Q(x))\}$  is open in  $E$ ;
- (iii)  $G_0 = \{x \in E : x \notin S(x)\}$  is open in  $E$ ;
- (iv) for each  $x \in E$ ,  $S(x)$  is nonempty  $\Gamma$ -convex,  $S^{-1}(y)$  is open for all  $y \in E$ ;
- (v) for each  $(x_0, y_0) \in E \times E$  with  $x_0 \in T(y_0)$  such that  $y_0 \notin S(x_0)$ .

*Then there exists  $\tilde{x} \in S(\tilde{x})$  such that  $(\alpha)z \in B(\tilde{x}), \rho(F(\tilde{x}, y, z), Q(\tilde{x}))$  for any  $y \in S(\tilde{x})$ .*

*Proof* For any  $x \in E$ , define  $A : E \rightrightarrows E$  by

$$A(x) = \{y \in E : (\bar{\alpha})z \in B(x), \rho^c(F(x, y, z), Q(x))\}.$$



From the definition of  $A(x)$ , one has

$$A^{-1}(y) = \{x \in E : (\bar{\alpha})z \in B(x), \rho^c(F(x, y, z), Q(x))\}.$$

Define  $P : E \rightrightarrows E$  by

$$P(x) = \begin{cases} S(x) \cap A(x), & x \in E \setminus G_0; \\ S(x), & x \in G_0. \end{cases} \quad (1)$$

Let  $M(y) = E \setminus P^{-1}(y)$ . We show that  $M(y)$  is closed for all  $y \in E$ . In fact, it follows from (1) that

$$\begin{aligned} P^{-1}(y) &= \{x \in E \setminus G_0 : y \in S(x) \cap A(x)\} \cup \{x \in G_0 : y \in S(x)\} \\ &= \{x \in E \setminus G_0 : x \in S^{-1}(y) \cap A^{-1}(y)\} \cup \{x \in G_0 : x \in S^{-1}(y)\} \\ &= \{(E \setminus G_0) \cap S^{-1}(y) \cap A^{-1}(y)\} \cup \{G_0 \cap S^{-1}(y)\} \\ &= S^{-1}(y) \cap (G_0 \cup A^{-1}(y)). \end{aligned}$$

Since  $S^{-1}(y)$ ,  $A^{-1}(y)$ , and  $G_0$  are open, we know that  $P^{-1}(y)$  is open and so  $M(y)$  is closed.

We show that  $M$  is a KKM mapping with respect to  $T$ . Suppose that  $M$  is not a KKM mapping with respect to  $T$ . Then there exist a finite subset  $N$  and a point  $x_0 \in E$  such that  $x_0 \in T(\Gamma_N) \setminus M(N)$ . This shows that there exists a point  $y_0 \in \Gamma_N$  such that  $x_0 \in T(y_0)$ ,  $x_0 \in P^{-1}(y)$  for any  $y \in N$ , and so  $N \subset P(x_0) \subset S(x_0)$ . Since  $S(x_0)$  is  $\Gamma$ -convex and  $N \in \langle S(x_0) \rangle$ , we know that  $y_0 \in \Gamma_N \subset S(x_0)$ , which is a contradiction. It follows that  $M$  is a KKM mapping with respect to  $T$ .

It follows from Lemma 2.4 that  $M$  has finite intersection property. From the facts that  $M(y) \subset E$  is closed and  $E$  is compact, we know that  $M(y)$  is compact for any  $y \in E$  and so

$$\bigcap_{y \in E} M(y) \neq \emptyset.$$

Thus, there exists a point  $\tilde{x} \in E$  such that

$$\tilde{x} \in \bigcap_{y \in E} M(y) = E \setminus \bigcup_{y \in E} P^{-1}(y).$$

This implies that  $\tilde{x} \notin P^{-1}(y)$  for all  $y \in E$  and so  $P(\tilde{x}) = \emptyset$ .

If  $\tilde{x} \in G_0$ , then it is easy to see that  $S(\tilde{x}) = P(\tilde{x}) = \emptyset$ , which is a contradiction. Therefore, we have

$$\tilde{x} \in E \setminus G_0 \quad \text{with } S(\tilde{x}) \cap A(\tilde{x}) = P(\tilde{x}) = \emptyset$$

and so

$$\tilde{x} \in S(\tilde{x}) \quad \text{and} \quad y \notin A(\tilde{x}), \forall y \in S(\tilde{x}),$$

that is,  $\tilde{x} \in S(\tilde{x})$ ,  $(\alpha)z \in B(\tilde{x})$ ,  $\rho(F(\tilde{x}, y, z), C(\tilde{x}))$  for all  $y \in S(\tilde{x})$ . This completes the proof.  $\square$

**Remark 3.1** By Lemma 2.1, it is easy to see that the condition (iii) can be replaced by the following condition:

(iii)'  $S : E \rightrightarrows E$  is a u.s.c. set-valued mapping.

Next we give some existence theorems in connection with the solution of the vector quasi-equilibrium problems (GVQEP1)-(GVQEP8).

**Theorem 3.2** *Assume that the conditions (i), (iii), (iv), and (v) in Theorem 3.1 are satisfied. Moreover, suppose that*

- (a) *for each  $y \in E$ ,  $F(\cdot, y, \cdot)$  is l.s.c. and  $C$  is closed;*
- (b)  *$B$  is l.s.c.*

*Then there exists  $\tilde{x} \in E$  such that  $\tilde{x} \in S(\tilde{x})$  and  $F(\tilde{x}, y, z) \subset C(\tilde{x})$  for all  $y \in S(\tilde{x})$  and  $z \in B(\tilde{x})$ .*

*Proof* Let

$$A(x) = \{y \in E : \exists z \in B(x), F(x, y, z) \not\subseteq C(x)\}.$$

We show that

$$A^{-1}(y) = \{x \in E : \exists z \in B(x), F(x, y, z) \not\subseteq C(x)\}$$

is open. Let  $\{x_\alpha\} \subseteq E \setminus A^{-1}(y)$  be a net with  $x_\alpha \rightarrow x_0$ . Then

$$F(x_\alpha, y, z') \subseteq C(x_\alpha), \quad \forall z' \in B(x_\alpha).$$

Since  $B$  and  $F(\cdot, y, \cdot)$  are l.s.c., by Lemma 2.3, for any  $z \in B(x_0)$  and  $v \in F(x_0, y, z)$ , there exist  $z_\alpha \in B(x_\alpha)$  and  $v_\alpha \in F(x_\alpha, y, z_\alpha)$  such that  $z_\alpha \rightarrow z$  and  $v_\alpha \rightarrow v$ . Now the closedness of  $C$  with  $v_\alpha \in C(x_\alpha)$  shows that  $v \in C(x)$  and so  $F(x_0, y, z) \subseteq C(x)$  for any  $z \in B(x_0)$ . This shows that  $x_0 \in E \setminus A^{-1}(y)$  and so  $E \setminus A^{-1}(y)$  is closed. Thus,  $A^{-1}(y)$  is open. It follows from Theorem 3.1 that there exists  $\tilde{x} \in E$  such that  $\tilde{x} \in S(\tilde{x})$  and  $F(\tilde{x}, y, z) \subseteq C(\tilde{x})$  for all  $y \in S(\tilde{x})$  and  $z \in B(\tilde{x})$ . This completes the proof.  $\square$

**Remark 3.2** Theorem 3.2 can be considered as a generalization of Theorem 3.3 in [4] under different conditions from the topological vector space to the abstract convex space.

**Corollary 3.1** *Assume that the conditions (i), (iii), (iv), and (v) in Theorem 3.1 are satisfied with  $B = S$ . Suppose that, for each  $y \in E$ ,  $F(\cdot, y)$  is l.s.c. and  $C$  is closed. Then there exists  $\tilde{x} \in E$  such that  $\tilde{x} \in S(\tilde{x})$  and  $F(\tilde{x}, y) \subseteq C(\tilde{x})$  for all  $y \in S(\tilde{x})$ .*

*Proof* The proof is similar to that of Theorem 3.2 and so we omit it here.  $\square$

**Remark 3.3** When  $S(x) = E$  for all  $x \in E$ , Corollary 3.1 was given by Theorem 1 of Yang and Huang [17] under quite different conditions.

**Theorem 3.3** *Assume that the conditions (i), (iii), (iv), and (v) in Theorem 3.1 are satisfied. Moreover, suppose that*

- (a) *for each  $y \in E$ ,  $F(\cdot, y, \cdot)$  is l.s.c. and  $C$  is closed;*

(b)  $B$  is u.s.c. and  $B(x)$  is compact for each  $x \in E$ .  
 Then there exist  $\tilde{x} \in E$  and  $\tilde{z} \in B(\tilde{x})$  such that  $\tilde{x} \in S(\tilde{x})$  and  $F(\tilde{x}, y, \tilde{z}) \subseteq C(\tilde{x})$  for all  $y \in S(\tilde{x})$ .

*Proof* Let

$$A(x) = \{y \in E : \forall z \in B(x), F(x, y, z) \not\subseteq C(x)\}.$$

We first show that

$$A^{-1}(y) = \{x \in E : \forall z \in B(x), F(x, y, z) \not\subseteq C(x)\}$$

is open. Let  $\{x_\alpha\} \subseteq E \setminus A^{-1}(y)$  be a net with  $x_\alpha \rightarrow x_0$ . Then there exists  $z_\alpha \in B(x_\alpha)$  such that  $F(x_\alpha, y, z_\alpha) \subseteq C(x_\alpha)$ . Since  $B$  is u.s.c. with compact values, by Lemma 2.2, there exists a subset net of  $\{z_\alpha\}$ , denoted again by  $\{z_\alpha\}$ , such that  $z_\alpha \rightarrow z_0 \in B(x_0)$ . The fact that  $F(\cdot, y, \cdot)$  is l.s.c. together with Lemma 2.3 shows that, for any  $v \in F(x_0, y, z_0)$ , there exists  $v_\alpha \in F(x_\alpha, y, z_\alpha)$  such that  $v_\alpha \rightarrow v$ . Since  $v_\alpha \in C(x_\alpha)$  and  $C$  is closed, we know that  $v \in C(x_0)$  and so  $F(x_0, y, z_0) \subseteq C(x_0)$  for some  $z_0 \in B(x_0)$ . This implies that  $x_0 \in E \setminus A^{-1}(y)$  and so  $E \setminus A^{-1}(y)$  is closed. Thus,  $A^{-1}(y)$  is open. It follows from Theorem 3.1 that there exist  $\tilde{x} \in E$  and  $\tilde{z} \in B(\tilde{x})$  such that  $\tilde{x} \in S(\tilde{x})$  and  $F(\tilde{x}, y, \tilde{z}) \subseteq C(\tilde{x})$  for any  $y \in S(\tilde{x})$ . This completes the proof.  $\square$

**Remark 3.4** When  $S(x) = E$  for all  $x \in E$ , the existence of the solutions for generalized vector quasi-equilibrium was studied in Theorem 3.1 of [7] in real Hausdorff topological vector spaces.

**Theorem 3.4** Assume that the conditions (i), (iii), (iv), and (v) in Theorem 3.1 are satisfied. Moreover, suppose that

- (a) for each  $y \in E$ ,  $F(\cdot, y, \cdot)$  is l.s.c.,  $C(x)$  is a set-valued mapping with a nonempty interior for each  $x \in E$ , the mapping  $W : E \rightrightarrows V$ , defined by  $W(x) = V \setminus (-\text{int } C(x))$ , is closed;
- (b)  $B$  is l.s.c.

Then there exists  $\tilde{x} \in E$  such that  $\tilde{x} \in S(\tilde{x})$  and  $F(\tilde{x}, y, z) \cap (-\text{int } C(\tilde{x})) = \emptyset$  for all  $y \in S(\tilde{x})$  and  $z \in B(\tilde{x})$ .

*Proof* Let

$$A(x) = \{y \in E : \exists z \in B(x), F(x, y, z) \cap (-\text{int } C(x)) \neq \emptyset\}.$$

We prove that

$$A^{-1}(y) = \{x \in E : \exists z \in B(x), F(x, y, z) \cap (-\text{int } C(x)) \neq \emptyset\}$$

is open. Let  $\{x_\alpha\} \subseteq E \setminus A^{-1}(y)$  be a net with  $x_\alpha \rightarrow x_0$ . Then

$$F(x_\alpha, y, z') \cap (-\text{int } C(x_\alpha)) = \emptyset, \quad \forall z' \in B(x_\alpha)$$

and so

$$F(x_\alpha, y, z') \subseteq W(x_\alpha) = V \setminus (-\text{int } C(x_\alpha)).$$

Similar to the proof of Theorem 3.2, we get

$$F(x_0, y, z) \subseteq W(x_0) = V \setminus (-\operatorname{int} C(x_0)).$$

This shows that  $x_0 \in E \setminus A^{-1}(y)$  and so  $E \setminus A^{-1}(y)$  is closed. Thus,  $A^{-1}(y)$  is open. It follows from Theorem 3.1 that there exists  $\tilde{x} \in E$  such that  $\tilde{x} \in S(\tilde{x})$  and

$$F(\tilde{x}, y, z) \cap (-\operatorname{int} C(\tilde{x})) = \emptyset, \quad \forall y \in S(\tilde{x}), \forall z \in B(\tilde{x}).$$

This completes the proof.  $\square$

**Remark 3.5** Theorem 3.4 can be considered as a generalization of Theorem 3.2 in [4] under different conditions from the topological vector space to the abstract convex space.

**Corollary 3.2** Assume that the conditions (i), (iii), (iv), and (v) in Theorem 3.1 are satisfied with  $S = B$ . Moreover, suppose that

- (a) for each  $y \in E$ ,  $F(\cdot, y)$  is l.s.c.,  $C(x)$  has a nonempty interior for each  $x \in E$ , the map  $W : E \rightrightarrows V$ , defined by  $W(x) = V \setminus (-\operatorname{int} C(x))$ , is closed.

Then there exists  $\tilde{x} \in E$  such that  $\tilde{x} \in S(\tilde{x})$  and  $F(\tilde{x}, y) \cap (-\operatorname{int} C(\tilde{x})) = \emptyset$  for all  $y \in S(\tilde{x})$ .

*Proof* The proof is similar to that of Theorem 3.4 and so we omit it here.  $\square$

**Remark 3.6** When  $S(x) = E$  for all  $x \in E$ , Corollary 3.2 was given by Theorem 2 of Yang and Huang [17] under quite different conditions.

**Theorem 3.5** Suppose the conditions (i), (iii), (iv), and (v) in Theorem 3.1 are satisfied. Moreover, suppose that

- (a) for each  $y \in E$ ,  $F(\cdot, y, \cdot)$  is l.s.c.,  $C(x)$  has a nonempty interior for each  $x \in E$ , and the mapping  $W : E \rightrightarrows V$ , defined by  $W(x) = V \setminus (-\operatorname{int} C(x))$ , is closed;  
(b)  $B$  is u.s.c. and  $B(x)$  is compact for each  $x \in E$ .

Then there exist  $\tilde{x} \in E$  and  $\tilde{z} \in B(\tilde{x})$  such that  $\tilde{x} \in S(\tilde{x})$  and  $F(\tilde{x}, y, \tilde{z}) \cap (-\operatorname{int} C(\tilde{x})) = \emptyset$  for all  $y \in S(\tilde{x})$ .

*Proof* Let

$$A(x) = \{y \in E : \forall z \in B(x), F(x, y, z) \cap (-\operatorname{int} C(x)) \neq \emptyset\}.$$

We show that

$$A^{-1}(y) = \{x \in E : \forall z \in B(x), F(x, y, z) \cap (-\operatorname{int} C(x)) \neq \emptyset\}$$

is open. Let  $\{x_\alpha\} \subseteq E \setminus A^{-1}(y)$  be a net with  $x_\alpha \rightarrow x_0$ . Then

$$F(x_\alpha, y, z_\alpha) \cap (-\operatorname{int} C(x_\alpha)) = \emptyset$$

for some  $z_\alpha \in B(x_\alpha)$ , that is,

$$F(x_\alpha, y, z_\alpha) \subseteq V \setminus (-\operatorname{int} C(x_\alpha)).$$

Using similar arguments to the proof of Theorem 3.3, we have

$$F(x_0, y, z_0) \subseteq W(x_0) = V \setminus (-\text{int } C(x_0))$$

for some  $z_0 \in B(x_0)$ . This shows that  $x_0 \in E \setminus A^{-1}(y)$  and so  $E \setminus A^{-1}(y)$  is closed. Thus,  $A^{-1}(y)$  is open. It follows from Theorem 3.1 that there exist  $\tilde{x} \in E$  and  $\tilde{z} \in B(\tilde{x})$  such that  $\tilde{x} \in S(\tilde{x})$  and

$$F(\tilde{x}, y, \tilde{z}) \cap (-\text{int } C(\tilde{x})) = \emptyset, \quad \forall y \in S(\tilde{x}).$$

This completes the proof.  $\square$

**Remark 3.7** When  $E$  is a nonempty convex compact of a topological vector space, Li and Li [27] studied the existence of solutions for (GVQEP4).

**Theorem 3.6** Assume that the conditions (i), (iii), (iv), and (v) are satisfied in Theorem 3.1. Moreover, suppose that

- (a) for each  $y \in E$ ,  $F(\cdot, y, \cdot)$  is u.s.c. with compact valued on  $E \times E \times E$  and  $C(x)$  has a nonempty interior for each  $x \in E$ , the mapping  $W : E \rightrightarrows V$ , defined by  $W(x) = V \setminus (-\text{int } C(x))$ , is closed;
- (b)  $B$  is l.s.c.

Then there exists  $\tilde{x} \in E$  such that  $\tilde{x} \in S(\tilde{x})$  and  $F(\tilde{x}, y, z) \not\subseteq -\text{int } C(\tilde{x})$  for all  $y \in S(\tilde{x})$  and  $z \in B(\tilde{x})$ .

*Proof* Let

$$A(x) = \{y \in E : \exists z \in B(x), F(x, y, z) \subseteq -\text{int } C(x)\}.$$

We prove that

$$A^{-1}(y) = \{x \in E : \exists z \in B(x), F(x, y, z) \subseteq -\text{int } C(x)\}$$

is open. Let  $x_\alpha \in E \setminus A^{-1}(y)$  be a net with  $x_\alpha \rightarrow x_0$ . Then

$$F(x_\alpha, y, z') \not\subseteq -\text{int } C(x_\alpha)$$

for any  $z' \in B(x_\alpha)$  and so there exists  $v_\alpha \in V$  such that

$$v_\alpha \in F(x_\alpha, y, z') \setminus (-\text{int } C(x_\alpha)).$$

Since  $B$  is l.s.c., by Lemma 2.3, for any  $z \in B(x_0)$ , there exists  $z_\alpha \in B(x_\alpha)$  such that  $z_\alpha \rightarrow z$ . Since  $F(\cdot, y, \cdot)$  is u.s.c. with compact valued, by Lemma 2.2, there exists a subnet of  $\{v_\alpha\}$ , denoted again by  $\{v_\alpha\}$ , such that  $v_\alpha \rightarrow v_0 \in F(x_0, y, z)$ . On the other hand, the fact that  $v_\alpha \notin -\text{int } C(x_\alpha)$  shows that  $v_\alpha \in W(x_\alpha)$ . Now the closedness of  $W$  shows that  $v_0 \in W(x_0)$  and so  $v_0 \notin -\text{int } C(x_0)$ . Thus  $F(x_0, y, z) \not\subseteq -\text{int } C(x_0)$  for any  $z \in B(x_0)$ . This implies that  $x_0 \in E \setminus A^{-1}(y)$  and so  $E \setminus A^{-1}(y)$  is closed. Thus,  $A^{-1}(y)$  is open. It follows from Theorem 3.1 that there exists  $\tilde{x} \in E$  such that  $\tilde{x} \in S(\tilde{x})$  and  $F(\tilde{x}, y, z) \not\subseteq -\text{int } C(\tilde{x})$  for all  $y \in S(\tilde{x})$  and  $z \in B(\tilde{x})$ . This completes the proof.  $\square$

**Corollary 3.3** Assume that the conditions (i), (iii), (iv), and (v) in Theorem 3.1 are satisfied with  $S = B$ . Moreover, suppose that

- (a) for each  $y \in E$ ,  $F(\cdot, y)$  is u.s.c. with compact valued on  $E \times E$  and  $C(x)$  has a nonempty interior for each  $x \in E$ , the mapping  $W : E \rightrightarrows V$ , defined by  $W(x) = V \setminus (-\text{int } C(x))$ , is closed.

Then there exists  $\tilde{x} \in E$  such that  $\tilde{x} \in S(\tilde{x})$  and  $F(\tilde{x}, y) \not\subseteq -\text{int } C(\tilde{x})$  for all  $y \in S(\tilde{x})$ .

*Proof* The proof is similar to that of Theorem 3.6 and so we omit it here.  $\square$

**Remark 3.8** When  $S(x) = E$  for all  $x \in E$ , Corollary 3.3 was given by Theorem 4 of Yang and Huang [17] under quite different conditions.

**Theorem 3.7** Assume that the conditions (i), (iii), (iv), and (v) are satisfied in Theorem 3.1. Moreover, suppose that

- (a) for each  $y \in E$ ,  $F(\cdot, y, \cdot)$  is u.s.c. with compact valued on  $E \times E \times E$  and  $C(x)$  has a nonempty interior for each  $x \in E$ , the mapping  $W : E \rightrightarrows V$ , defined by  $W(x) = V \setminus (-\text{int } C(x))$ , is closed.

- (b)  $B$  is u.s.c. and  $B(x)$  is compact for each  $x \in E$ .

Then there exist  $\tilde{x} \in E$  and  $\tilde{z} \in B(\tilde{x})$  such that  $\tilde{x} \in S(\tilde{x})$  and  $F(\tilde{x}, y, \tilde{z}) \not\subseteq -\text{int } C(\tilde{x})$  for all  $y \in S(\tilde{x})$ .

*Proof* Let

$$A(x) = \{y \in E : \forall z \in B(x), F(x, y, z) \subseteq -\text{int } C(x)\}.$$

We prove that

$$A^{-1}(y) = \{x \in E : \forall z \in B(x), F(x, y, z) \subseteq -\text{int } C(x)\}$$

is open. Let  $x_\alpha \in E \setminus A^{-1}(y)$  be a net with  $x_\alpha \rightarrow x_0$ . Then  $F(x_\alpha, y, z_\alpha) \not\subseteq -\text{int } C(x_\alpha)$  for some  $z_\alpha \in B(x_\alpha)$  and so there exists  $v_\alpha \in V$  such that

$$v_\alpha \in F(x_\alpha, y, z_\alpha) \setminus (-\text{int } C(x_\alpha)).$$

Since  $B$  is u.s.c. with compact valued, by Lemma 2.2, there exists a subnet of  $\{z_\alpha\}$ , denoted again by  $\{z_\alpha\}$ , such that  $z_\alpha \rightarrow z_0 \in B(x_0)$ . The fact that  $F(\cdot, y, \cdot)$  is u.s.c. with compact valued together with Lemma 2.2 shows that there exists a subset net of  $\{v_\alpha\}$ , denoted again by  $\{v_\alpha\}$ , such that  $v_\alpha \rightarrow v_0 \in F(x_0, y, z_0)$ . On the other hand, it is easy to see that  $v_\alpha \in W(x_\alpha)$ . Since  $W$  is closed, we know that  $v_0 \in W(x_0)$  and so  $v_0 \notin -\text{int } C(x_0)$ . Thus  $F(x_0, y, z_0) \not\subseteq -\text{int } C(x_0)$  for some  $z_0 \in B(x_0)$  and so  $x_0 \in E \setminus A^{-1}(y)$ . This implies that  $E \setminus A^{-1}(y)$  is closed and so  $A^{-1}(y)$  is open. It follows from Theorem 3.1 that there exist  $\tilde{x} \in E$  and  $\tilde{z} \in B(\tilde{x})$  such that  $\tilde{x} \in S(\tilde{x})$  and  $F(\tilde{x}, y, \tilde{z}) \not\subseteq -\text{int } C(\tilde{x})$  for all  $y \in S(\tilde{x})$ . This completes the proof.  $\square$

**Remark 3.9** Theorem 3.7 can be considered as a generalization of Theorem 3.1 in [2, 4] under different conditions from the topological vector space to the abstract convex space.

**Remark 3.10** When  $E$  is a nonempty convex compact of topological vector space, Li and Li [27] studied the existence of solutions for (GVQEP6).

**Theorem 3.8** *Suppose the conditions (i), (iii), (iv), and (v) in Theorem 3.1 are satisfied. Moreover, assume that*

- (a) *for each  $y \in E$ ,  $F(\cdot, y, \cdot)$  is u.s.c. with compact valued on  $E \times E \times E$  and  $C$  is closed;*
- (b)  *$B$  is l.s.c.*

*There exists  $\tilde{x} \in E$  such that  $\tilde{x} \in S(\tilde{x})$  and  $F(\tilde{x}, y, z) \cap C(\tilde{x}) \neq \emptyset$  for all  $y \in S(\tilde{x})$  and  $z \in B(x)$ .*

*Proof* Let

$$A(x) = \{y \in E : \exists z \in B(x), F(x, y, z) \cap C(x) = \emptyset\}.$$

We show that

$$A^{-1}(y) = \{x \in E : \exists z \in B(x), F(x, y, z) \cap C(x) = \emptyset\}$$

is open. Let  $\{x_\alpha\} \subseteq E \setminus A^{-1}(y)$  be a net with  $x_\alpha \rightarrow x_0$ . Then

$$F(x_\alpha, y, z') \cap C(x_\alpha) \neq \emptyset, \quad \forall z' \in B(x_\alpha).$$

It follows that there exists  $v_\alpha \in F(x_\alpha, y, z') \cap C(x_\alpha)$ . Since  $B$  is l.s.c., by Lemma 2.3, there exists  $z_\alpha \in B(x_\alpha)$  such that  $z_\alpha \rightarrow z$  for any  $z \in B(x_0)$ . By the fact that  $F(\cdot, y, \cdot)$  is u.s.c. with compact valued, there exists a subset of  $\{v_\alpha\}$ , denoted again by  $\{v_\alpha\}$ , such that  $v_\alpha \rightarrow v_0 \in F(x_0, y, z)$ . Since  $v_\alpha \in C(x_\alpha)$  and  $C$  is closed, we know that  $v_0 \in C(x_0)$  and so  $v_0 \in F(x_0, y, z) \cap C(x_0)$ . Thus,

$$F(x_0, y, z) \cap C(x_0) \neq \emptyset, \quad \forall z \in B(x_0).$$

This shows that  $x_0 \in E \setminus A^{-1}(y)$  and so  $E \setminus A^{-1}(y)$  is closed. Thus,  $A^{-1}(y)$  is open. By Theorem 3.1, there exists  $\tilde{x} \in E$  such that  $\tilde{x} \in S(\tilde{x})$  and

$$F(\tilde{x}, y, z) \cap C(\tilde{x}) \neq \emptyset, \quad \forall y \in S(\tilde{x}), \forall z \in B(\tilde{x}).$$

This completes the proof. □

**Corollary 3.4** *Assume that the conditions (i), (iii), (iv), and (v) in Theorem 3.1 are satisfied with  $S = B$ . Moreover, suppose that, for each  $y \in E$ ,  $F(\cdot, y, \cdot)$  is u.s.c. with compact valued on  $E \times E$  and  $C$  is closed. Then there exists  $\tilde{x} \in E$  such that  $\tilde{x} \in S(\tilde{x})$  and  $F(\tilde{x}, y) \cap C(\tilde{x}) \neq \emptyset$  for all  $y \in S(\tilde{x})$ .*

*Proof* The proof is similar to that of Theorem 3.8 and so we omit it here. □

**Remark 3.11** When  $S(x) = E$  for all  $x \in E$ , Corollary 3.4 was given by Theorem 3 of Yang and Huang [17] under some different conditions.

**Theorem 3.9** *Suppose the conditions (i), (iii), (iv), and (v) are satisfied in Theorem 3.1. Moreover, assume that*

- (a) *for each  $y \in E$ ,  $F(\cdot, y, \cdot)$  is u.s.c. with compact valued on  $E \times E \times E$  and  $C$  is closed;*
- (b)  *$B$  is u.s.c. and  $B(x)$  is compact for each  $x \in E$ .*

*There exist  $\tilde{x} \in E$  and  $\tilde{z} \in B(\tilde{x})$  such that  $\tilde{x} \in S(\tilde{x})$  and  $F(\tilde{x}, y, \tilde{z}) \cap C(\tilde{x}) \neq \emptyset$  for all  $y \in S(\tilde{x})$ .*

*Proof* Let

$$A(x) = \{y \in E : \forall z \in B(x), F(x, y, z) \cap C(x) = \emptyset\}.$$

We prove that

$$A^{-1}(y) = \{x \in E : \forall z \in B(x), F(x, y, z) \cap C(x) = \emptyset\}$$

is open. Let  $\{x_\alpha\} \subseteq E \setminus A^{-1}(y)$  be a net with  $x_\alpha \rightarrow x_0$ . Then

$$F(x_\alpha, y, z_\alpha) \cap C(x_\alpha) \neq \emptyset$$

for some  $z_\alpha \in B(x_\alpha)$ , that is, there exists  $v_\alpha \in F(x_\alpha, y, z_\alpha) \cap C(x_\alpha)$ . Since  $B$  is u.s.c. and  $B(x)$  is compact, it follows from Lemma 2.2 that there exists a subset of  $\{z_\alpha\}$ , denoted again by  $\{z_\alpha\}$ , such that  $z_\alpha \rightarrow z_0 \in B(x_0)$ . Similar to the proof of Theorem 3.8, we can prove that  $F(x_0, y, z_0) \cap C(x_0) \neq \emptyset$  for some  $z_0 \in B(x_0)$ . This shows that  $x_0 \in E \setminus A^{-1}(y)$  and so  $E \setminus A^{-1}(y)$  is closed. Thus,  $A^{-1}(y)$  is open. It follows from Theorem 3.1 that there exist  $\tilde{x} \in E$  and  $\tilde{z} \in B(\tilde{x})$  such that  $\tilde{x} \in S(\tilde{x})$  and

$$F(\tilde{x}, y, \tilde{z}) \cap C(\tilde{x}) \neq \emptyset, \quad \forall y \in S(\tilde{x}).$$

This completes the proof.  $\square$

#### 4 Applications to the generalized semi-infinite programs

In this section, by the results presented in Section 3, we give some existence theorems of solutions to the generalized semi-infinite programs. Let  $L$  be a real topological vector space ordered by a closed convex pointed cone  $H \subseteq L$  with  $\text{int } H \neq \emptyset$  and  $h : X \rightrightarrows L$  be a u.s.c. mapping with compact values.

**Theorem 4.1** *Suppose that all the conditions of Theorem 3.2 are satisfied. Moreover, assume that  $F(\cdot, \cdot, \cdot)$  and  $S$  are l.s.c. Then there is a solution to the problem*

$$\text{wMin}_H h(K),$$

where

$$K = \{x \in E : x \in S(x), F(x, y, z) \subseteq C(x), \forall y \in S(x), \forall z \in B(x)\}.$$

*Proof* Theorem 3.2 shows that  $K \neq \emptyset$ . From Lemma 2.5, it is sufficient to show that  $h(K)$  is compact. Since  $h$  is u.s.c. and  $K \subseteq E$ , by Lemma 2.1, we only need to prove that  $K$  is closed. Let  $\{x_\alpha\} \subseteq K$  be a net with  $x_\alpha \rightarrow x_0$ . Then  $x_\alpha \in S(x_\alpha)$  and

$$F(x_\alpha, y', z') \subseteq C(x_\alpha), \quad \forall y' \in S(x_\alpha), \forall z' \in B(x_\alpha).$$

Since  $S$  and  $B$  are l.s.c., for any  $y \in S(x_0)$  and  $z \in B(x_0)$ , it follows from Lemma 2.3 that there exist  $y_\alpha \in S(x_\alpha)$  and  $z_\alpha \in B(x_\alpha)$  such that  $y_\alpha \rightarrow y$  and  $z_\alpha \rightarrow z$ . By the lower semi-continuity



of  $F$  and Lemma 2.3, for any  $v \in F(x_0, y, z)$ , there exists  $v_\alpha \in F(x_\alpha, y_\alpha, z_\alpha)$  such that  $v_\alpha \rightarrow v$ . Now the closedness of  $C$  with  $v_\alpha \in C(x_\alpha)$  shows that  $v \in C(x_0)$  and so  $F(x_0, y, z) \subseteq C(x_0)$  for all  $y \in S(x_0)$  and  $z \in B(x_0)$ . Moreover, the closedness of  $E \setminus G_0$  shows that  $x_0 \in S(x_0)$ . Thus,  $K$  is closed. This completes the proof.  $\square$

**Corollary 4.1** *Suppose that all the conditions of Corollary 3.1 are satisfied. Moreover, assume that  $F(\cdot, \cdot)$  and  $S$  are l.s.c. Then there is a solution to the problem*

$$\text{wMin}_H h(K),$$

where

$$K = \{x \in E : x \in S(x), F(x, y) \subseteq C(x), \forall y \in S(x)\}.$$

**Remark 4.1** When  $S(x) = E$  for all  $x \in E$ , Corollary 4.1 was given by Theorem 5 of Yang and Huang [17] under some different conditions.

**Theorem 4.2** *Suppose that all the conditions of Theorem 3.3 are satisfied. Moreover, assume that  $F(\cdot, \cdot, \cdot)$  and  $S$  are l.s.c. Then there is a solution to the problem*

$$\text{wMin}_H h(K),$$

where

$$K = \{x \in E : x \in S(x), \exists z \in B(x), F(x, y, z) \subseteq C(x), \forall y \in S(x)\}.$$

*Proof* Obviously, Theorem 3.3 shows that  $K \neq \emptyset$ . By Lemma 2.5, it is sufficient to prove that  $h(K)$  is compact. Since  $h$  is u.s.c. and  $K \subseteq E$ , from Lemma 2.1, we only need to show that  $K$  is closed. Let  $\{x_\alpha\} \subseteq K$  be a net with  $x_\alpha \rightarrow x_0$ . Then  $x_\alpha \in S(x_\alpha)$  and there exists  $z_\alpha \in B(x_\alpha)$  such that

$$F(x_\alpha, y', z_\alpha) \subseteq C(x_\alpha), \quad \forall y' \in S(x_\alpha).$$

Since  $B$  is a u.s.c. mapping with compact values, it follows from Lemma 2.2 that there exists a subnet of  $\{z_\alpha\}$ , denoted again by  $\{z_\alpha\}$ , such that  $z_\alpha \rightarrow z_0 \in B(x_0)$ . For any  $y \in S(x_0)$ , the lower semi-continuity of  $S$  together with Lemma 2.3 implies that there exists  $y_\alpha \in S(x_\alpha)$  such that  $y_\alpha \rightarrow y$ . For  $v \in F(x_0, y, z_0)$ , by the fact that  $F$  is l.s.c., it follows from Lemma 2.3 that there exists  $v_\alpha \in F(x_\alpha, y_\alpha, z_\alpha)$  such that  $v_\alpha \rightarrow v$ . Now the closedness of  $C$  with  $v_\alpha \in C(x_\alpha)$  shows that  $v \in C(x_0)$  and so there exists  $z_0 \in B(x_0)$  such that  $F(x_0, y, z) \subseteq C(x_0)$  for all  $y \in S(x_0)$ . Moreover, the closedness of  $E \setminus G_0$  shows that  $x_0 \in S(x_0)$ . Thus,  $K$  is closed. This completes the proof.  $\square$

**Theorem 4.3** *Suppose that all the conditions of Theorem 3.4 are satisfied. Moreover, assume that  $F(\cdot, \cdot, \cdot)$  and  $S$  are l.s.c. Then there is a solution to the problem*

$$\text{wMin}_H h(K),$$

where

$$K = \{x \in E : x \in S(x), F(x, y, z) \cap -\text{int } C(x) = \emptyset, \forall y \in S(x), \forall z \in B(x)\}.$$

*Proof* Theorem 3.4 shows that  $K \neq \emptyset$ . From Lemma 2.5, it is sufficient to show that  $h(K)$  is compact. Since  $h$  is u.s.c. and  $K \subseteq E$ , by Lemma 2.1, we only need to show that  $K$  is closed. Let  $\{x_\alpha\} \subseteq K$  be a net with  $x_\alpha \rightarrow x_0$ . Then  $x_\alpha \in S(x_\alpha)$ ,

$$F(x_\alpha, y', z') \cap -\text{int } C(x) = \emptyset, \quad \forall y' \in S(x_\alpha), \forall z' \in B(x_\alpha)$$

and so

$$F(x_\alpha, y', z') \subseteq W(x_\alpha), \quad \forall y' \in S(x_\alpha), \forall z' \in B(x_\alpha).$$

Similar to the proof of Theorem 4.1, we have  $x_0 \in S(x_0)$ ,

$$F(x_0, y, z) \subseteq W(x_0), \quad \forall y \in S(x_0), \forall z \in B(x_0)$$

and so

$$F(x_0, y, z) \cap -\text{int } C(x_0) = \emptyset, \quad \forall y \in S(x_0), \forall z \in B(x_0).$$

Thus,  $K$  is closed. This completes the proof.  $\square$

**Corollary 4.2** *Suppose that all the conditions of Corollary 3.2 are satisfied. Moreover, assume that  $F(\cdot, \cdot)$  and  $S$  are l.s.c. Then there is a solution to the problem*

$$\text{wMin}_H h(K),$$

where

$$K = \{x \in E : x \in S(x), F(x, y) \cap C(x) = \emptyset, \forall y \in S(x)\}.$$

**Remark 4.2** When  $S(x) = E$  for all  $x \in E$ , Corollary 4.2 was given by Theorem 6 of Yang and Huang [17] under some different conditions.

**Theorem 4.4** *Suppose that all the conditions of Theorem 3.5 are satisfied. Moreover, assume that  $F(\cdot, \cdot, \cdot)$  and  $S$  are l.s.c. Then there is a solution to the problem*

$$\text{wMin}_H h(K),$$

where

$$K = \{x \in E : x \in S(x), \exists z \in B(x), F(x, y, z) \cap -\text{int } C(x) = \emptyset, \forall y \in S(x)\}.$$

*Proof* It follows from Theorem 3.5 that  $K \neq \emptyset$ . From Lemma 2.5, it is sufficient to show that  $h(K)$  is compact. Since  $h$  is u.s.c. and  $K \subseteq E$ , by Lemma 2.1, we only need to show  $K$  is closed. Let  $\{x_\alpha\} \subseteq K$  be a net with  $x_\alpha \rightarrow x_0$ . Then  $x_\alpha \in S(x_\alpha)$  and there exists  $z_\alpha \in B(x_\alpha)$  such that

$$F(x_\alpha, y', z_\alpha) \cap -\text{int } C(x_\alpha) = \emptyset, \quad \forall y' \in S(x_\alpha).$$

Thus, there exists  $z_\alpha \in B(x_\alpha)$  such that

$$F(x_\alpha, y', z_\alpha) \subseteq C(x_\alpha), \quad \forall y' \in S(x_\alpha).$$

Similar to the proof of Theorem 4.2, we know that  $x_0 \in S(x_0)$  and there exists  $z_0 \in B(x_0)$  such that

$$F(x_0, y, z_0) \subseteq W(x_0), \quad \forall y \in S(x_0).$$

Thus,  $x_0 \in S(x_0)$  and there exists  $z_0 \in B(x_0)$  such that

$$F(x_0, y, z_0) \cap -\text{int } C(x_0) = \emptyset, \quad \forall y \in S(x_0).$$

It follows that  $K$  is closed. This completes the proof.  $\square$

**Theorem 4.5** *Suppose that all the conditions of Theorem 3.6 are satisfied. Moreover, assume that  $F(\cdot, \cdot, \cdot)$  is a u.s.c. mapping with compact values and  $S$  is l.s.c. Then there is a solution to the problem*

$$\text{wMin}_H h(K),$$

where

$$K = \{x \in E : x \in S(x), F(x, y, z) \not\subseteq -\text{int } C(x), \forall y \in S(x), \forall z \in B(x)\}.$$

*Proof* Theorem 3.6 shows that  $K \neq \emptyset$ . From Lemma 2.5, it is sufficient to show that  $h(K)$  is compact. Since  $h$  is u.s.c. and  $K \subseteq E$ , by Lemma 2.1, we only need to prove that  $K$  is closed. Let  $\{x_\alpha\} \subseteq K$  be a net with  $x_\alpha \rightarrow x_0$ . Then  $x_\alpha \in S(x_\alpha)$ ,

$$F(x_\alpha, y', z') \not\subseteq -\text{int } C(x_\alpha), \quad \forall y' \in S(x_\alpha), \forall z' \in B(x_\alpha)$$

and so there exists  $v_\alpha \in V$  such that

$$v_\alpha \in F(x_\alpha, y', z') \setminus (-\text{int } C(x_\alpha)).$$

By the lower semi-continuity of  $S$  and  $B$ , for any  $y \in S(x_0)$  and  $z \in B(x_0)$ , it follows from Lemma 2.3 that there exist  $y_\alpha \in S(x_\alpha)$  and  $z_\alpha \in B(x_\alpha)$  such that  $y_\alpha \rightarrow y$  and  $z_\alpha \rightarrow z$ . Since  $F(\cdot, \cdot, \cdot)$  is a u.s.c. mapping with compact values, Lemma 2.2 shows that there exists a subnet of  $\{v_\alpha\}$ , denoted again by  $\{v_\alpha\}$ , such that  $v_\alpha \rightarrow v_0 \in F(x_0, y, z)$ . On the other hand, the fact

that  $v_\alpha \notin -\text{int } C(x_\alpha)$  shows that  $v_\alpha \in W(x_\alpha)$ . Now the closedness of  $W$  shows that  $v_0 \in W(x_0)$  and so  $v_0 \notin -\text{int } C(x_0)$ . Moreover, the closedness of  $E \setminus G_0$  shows that  $x_0 \in S(x_0)$ . Thus,

$$F(x_0, y, z) \not\subseteq -\text{int } C(x_0)$$

for all  $y \in S(x_0)$  and  $z \in B(x_0)$  and so  $K$  is closed. This completes the proof.  $\square$

**Corollary 4.3** *Suppose that all the conditions of Corollary 3.3 are satisfied. Moreover, assume that  $F(\cdot, \cdot)$  is u.s.c. and  $S$  is l.s.c. Then there is a solution to the problem*

$$\text{wMin}_H h(K),$$

where

$$K = \{x \in E : x \in S(x), F(x, y) \not\subseteq C(x), \forall y \in S(x)\}.$$

**Remark 4.3** When  $S(x) = E$  for all  $x \in E$ , Corollary 4.3 was given by Theorem 8 of Yang and Huang [17] under some different conditions.

**Theorem 4.6** *Suppose that all the conditions of Theorem 3.7 are satisfied. Moreover, assume that  $F(\cdot, \cdot, \cdot)$  is a u.s.c. mapping with compact values and  $S$  is l.s.c. Then there is a solution to the problem*

$$\text{wMin}_H h(K),$$

where

$$K = \{x \in E : x \in S(x), \exists z \in B(x), F(x, y, z) \not\subseteq -\text{int } C(x), \forall y \in S(x)\}.$$

*Proof* Theorem 3.7 shows that  $K \neq \emptyset$ . By Lemma 2.5, it is sufficient to prove that  $h(K)$  is compact. Since  $h$  is u.s.c. and  $K \subseteq E$ , from Lemma 2.1, we only need to show that  $K$  is closed. Let  $\{x_\alpha\} \subseteq K$  be a net with  $x_\alpha \rightarrow x_0$ . Then  $x_\alpha \in S(x_\alpha)$  and there exists  $z_\alpha \in B(x_\alpha)$  such that

$$F(x_\alpha, y', z_\alpha) \not\subseteq -\text{int } C(x_\alpha), \quad \forall y' \in S(x_\alpha).$$

Thus, there exists  $v_\alpha \in V$  such that

$$v_\alpha \in F(x_\alpha, y', z_\alpha) \setminus (-\text{int } C(x_\alpha)).$$

Since  $B$  is a u.s.c. mapping with compact values, it follows from Lemma 2.2 that there exists a subnet of  $\{z_\alpha\}$ , denoted again by  $\{z_\alpha\}$ , such that  $z_\alpha \rightarrow z_0 \in B(x_0)$ . By the lower semi-continuity of  $S$ , for any  $y \in S(x_0)$ , Lemma 2.3 shows that there exists  $y_\alpha \in S(x_\alpha)$  such that  $y_\alpha \rightarrow y$ . Since  $F(\cdot, \cdot, \cdot)$  is a u.s.c. mapping with compact values, Lemma 2.2 implies that there exists a subnet of  $\{v_\alpha\}$ , denoted again by  $\{v_\alpha\}$ , such that  $v_\alpha \rightarrow v_0 \in F(x_0, y, z_0)$ . Similar to the proof of Theorem 4.5, we can prove that  $K$  is closed. This completes the proof.  $\square$

**Theorem 4.7** *Suppose that all the conditions of Theorem 3.8 are satisfied. Moreover, assume that  $F(\cdot, \cdot, \cdot)$  is a u.s.c. mapping with compact values and  $S$  is l.s.c. Then there is a solution to the problem*

$$\text{wMin}_H h(K),$$

where

$$K = \{x \in E : x \in S(x), F(x, y, z) \cap C(x) \neq \emptyset, \forall y \in S(x), \forall z \in B(x)\}.$$

*Proof* Theorem 3.8 shows that  $K \neq \emptyset$ . From Lemma 2.5, it is sufficient to show that  $h(K)$  is compact. Since  $h$  is u.s.c. and  $K \subseteq E$ , by Lemma 2.1, we only need to show  $K$  is closed. Let  $\{x_\alpha\} \subseteq K$  be a net with  $x_\alpha \rightarrow x_0$ . Then  $x_\alpha \in S(x_\alpha)$ ,

$$F(x_\alpha, y', z') \cap C(x_\alpha) \neq \emptyset, \quad \forall y' \in S(x_\alpha), z' \in B(x_\alpha)$$

and so there exists  $v_\alpha \in V$  such that

$$v_\alpha \in F(x_\alpha, y', z') \cap C(x_\alpha).$$

By the lower semi-continuity of  $S$  and  $B$ , for any  $y \in S(x_0)$  and  $z \in B(x_0)$ , Lemma 2.3 shows that there exist  $y_\alpha \in S(x_\alpha)$  and  $z_\alpha \in B(x_\alpha)$  such that  $y_\alpha \rightarrow y$  and  $z_\alpha \rightarrow z$ . Since  $F(\cdot, \cdot, \cdot)$  is u.s.c. with compact values, by Lemma 2.2, there exists a subnet of  $\{v_\alpha\}$ , denoted again by  $\{v_\alpha\}$ , such that  $v_\alpha \rightarrow v_0 \in F(x_0, y, z)$ . Now the closedness of  $C$  with  $v_\alpha \in C(x_\alpha)$  shows that  $v_0 \in C(x_0)$  and so

$$F(x_0, y, z) \cap C(x_0) \neq \emptyset, \quad \forall y \in S(x_0), \forall z \in B(x_0).$$

Moreover, the closedness of  $E \setminus G_0$  shows that  $x_0 \in S(x_0)$ . Thus,  $K$  is closed. This completes the proof.  $\square$

**Corollary 4.4** *Suppose that all the conditions of Corollary 3.4 are satisfied. Moreover, assume that  $F(\cdot, \cdot)$  and  $S$  are l.s.c. Then there is a solution to the problem*

$$\text{wMin}_H h(K),$$

where

$$K = \{x \in E : x \in S(x), F(x, y) \cap C(x) \neq \emptyset, \forall y \in S(x)\}.$$

**Remark 4.4** When  $S(x) = E$  for all  $x \in E$ , Corollary 4.4 was given by Theorem 7 of Yang and Huang [17] under some different conditions.

**Theorem 4.8** *Suppose that all the conditions of Theorem 3.9 are satisfied. Moreover, assume that  $F(\cdot, \cdot, \cdot)$  is a u.s.c. mapping with compact values and  $S$  is l.s.c. Then there is a solution to the problem*

$$\text{wMin}_H h(K),$$

where

$$K = \{x \in E : x \in S(x), \exists z \in B(x), F(x, y, z) \cap C(x) \neq \emptyset, \forall y \in S(x)\}.$$

*Proof* Theorem 3.8 shows that  $K \neq \emptyset$ . From Lemma 2.5, it is sufficient to prove that  $h(K)$  is compact. Since  $h$  is u.s.c. and  $K \subseteq E$ , by Lemma 2.1, we only need to show  $K$  is closed. Let  $\{x_\alpha\} \subseteq K$  be a net with  $x_\alpha \rightarrow x_0$ . Then  $x_\alpha \in S(x_\alpha)$  and there exists  $z_\alpha \in B(x_\alpha)$  such that

$$F(x_\alpha, y', z_\alpha) \cap C(x_\alpha) \neq \emptyset, \quad \forall y' \in S(x_\alpha).$$

Thus, there exists  $v_\alpha \in V$  such that

$$v_\alpha \in F(x_\alpha, y', z_\alpha) \cap C(x_\alpha).$$

Since  $B$  is a u.s.c. mapping with compact values, it follows from Lemma 2.2 that there exists a subnet of  $\{z_\alpha\}$ , denoted again by  $\{z_\alpha\}$ , such that  $z_\alpha \rightarrow z_0 \in B(x_0)$ . By the lower semi-continuity of  $S$ , for any  $y \in S(x_0)$ , Lemma 2.3 implies that there exists  $y_\alpha \in S(x_\alpha)$  such that  $y_\alpha \rightarrow y$ . Since  $F(\cdot, \cdot, \cdot)$  is a u.s.c. mapping with compact values, by Lemma 2.2, there exists a subnet of  $\{v_\alpha\}$ , denoted again by  $\{v_\alpha\}$ , such that  $v_\alpha \rightarrow v_0 \in F(x_0, y, z_0)$ . Now the closedness of  $C$  with  $v_\alpha \in C(x_\alpha)$  shows that  $v_0 \in C(x_0)$  and so there exists  $z_0 \in B(x_0)$  such that

$$F(x_0, y, z_0) \cap C(x_0) \neq \emptyset, \quad \forall y \in S(x_0).$$

Moreover, the closedness of  $E \setminus G_0$  shows that  $x_0 \in S(x_0)$ . Therefore,  $K$  is closed. This completes the proof.  $\square$

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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